A linear kernel for planar total dominating set*

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Abstract

A *total dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex in V is adjacent to some vertex in D. Finding a total dominating set of minimum size is NP-complete on planar graphs and $W[2]$-complete on general graphs when parameterized by the solution size. By the meta-theorem of Bodlaender *et al.* [FOCS 2009], it follows that there *exists* a linear kernel for Total Dominating Set on graphs of bounded genus. Nevertheless, it is not clear how such a kernel can be effectively *constructed*, and how to obtain explicit reduction rules with reasonably small constants. Following the approach of Alber *et al.* [J. ACM 2004], we provide an explicit linear kernel for Total Dominating Set on planar graphs. This result complements several known constructive linear kernels on planar graphs for other domination problems such as Dominating Set, Edge Dominating Set, Efficient Dominating Set, or Connected Dominating Set.

Keywords: parameterized complexity, planar graphs, linear kernels, total domination.

1 Introduction

Motivation. The field of parameterized complexity deals with algorithms for decision problems whose instances consist of a pair (x, k), where k is a secondary measurement known as the *parameter*. A fundamental concept in this area is that of *kernelization*. A kernelization algorithm, or just *kernel*, for a parameterized problem takes an instance (x, k) of the problem and, in time polynomial in $|x| + k$, outputs an equivalent instance (x', k') such that $|x'|, k' \leq g(k)$ for some function g. The function g is called the *size* of the kernel and may be viewed as a measure of the "compressibility" of a problem using polynomial-time preprocessing rules. A natural problem in this context is to find polynomial or linear kernels for problems that admit such kernelization algorithms. For an introduction to parameterized complexity and kernelization see [7, 8, 25].

During the last decade, a plethora of results emerged on linear kernels for graph-theoretic problems restricted to *sparse* graph classes, that is, classes of graphs for which the number of edges depends linearly on the number of vertices. A pioneering result in this area is the linear kernel for Dominating Set on planar graphs by Alber *et al.* [1], which gave rise to an explosion of results on linear kernels on planar graphs and other sparse graph classes. Let us just mention some of the most important ones. Following the ideas of Alber *et al.* [1], Guo and Niedermeier [15] designed a general framework and showed that problems that satisfy a certain *distance property* have linear

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kernels on planar graphs. This result was subsumed by that of Bodlaender *et al.* [3] who provided a meta-theorem for problems to have a linear kernel on graphs of bounded genus. Later Fomin *et al.* [9] extended these results for *bidimensional* problems to an even larger graph class, namely, H-minor-free and apex-minor-free graphs. The most general result in this area is by Kim *et al.* [23], who provided linear kernels for *treewidth-bounding* problems on H-topological-minor-free graphs. (Note that in all these works, the problems are parameterized by the solution size.)

A common feature of these meta-theorems on sparse graphs is a *decomposition scheme* of the input graph that, loosely speaking, allows to deal with each part of the decomposition independently. For instance, the approach of [15], which strongly builds on [1], is to consider a so-called *region decomposition* of the input planar graph. The key point is that in an appropriately reduced YES-instance, there are $O(k)$ regions and each one has constant size, yielding the desired linear kernel. This idea was generalized in [3] to graphs on surfaces, where the role of regions is played by *protrusions*, which are graphs with small treewidth and small boundary. A crucial point is that while the reduction rules of [1, 15] are *problem-dependent*, those of [3] are *automated*, relying on a property called *finite integer index* (FII), which was introduced by Bodlaender and de Fluiter [4]. Loosely speaking, having FII guarantees that "large" protrusions of a graph can be replaced by "small" gadget graphs preserving equivalence of instances. FII is also of central importance to the approach of [9] (resp. [23]) on H-minor-free (resp. H-minor-topological-free) graphs. See [3, 9, 23] for more details.

Although of great theoretical importance, the aforementioned meta-theorems have two important drawbacks from a practical point of view. On the one hand, these results relying on FII guarantee the *existence* of a linear kernel, but nowadays it is still not clear how such a kernel can be effectively *constructed*. On the other hand, even if we knew how to construct such kernels, at the price of generality one cannot hope them to provide explicit reduction rules and small constants for a particular graph problem. Summarizing, as mentioned explicitly by Bodlaender *et al.* [3], these meta-theorems provide simple criteria to decide whether a problem admits a linear kernel on a graph class, but finding linear kernels with reasonably small constant factors for concrete problems remains a worthy investigation topic.

In this article we follow this research avenue and focus on the TOTAL DOMINATING SET problem on planar graphs. A *total dominating set* (or TDS for short) of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex in V is adjacent to some vertex in D (equivalently, a total dominating set is a dominating set inducing a subgraph without isolated vertices). Total domination was introduced by Cockayne, Dawes, and Hedetniemi [6] more than three decades ago and has remained a very active topic in graph theory since then (cf. [19, 21, 22, 27] for a few examples). More details and references can be found in the comprehensive survey of Henning [18] or the book of Haynes, Hedetniemi, and Slater [16]. Fig. 1 gives an example showing that, in particular, a minimal dominating set is not necessarily a subset of a minimal TDS, or vice versa.

From a (classical) complexity point of view, finding a TDS of minimum size is NP-complete on planar graphs [13, 28]. From a parameterized complexity perspective (see [7, 8, 25] for the missing definitions), TOTAL DOMINATING SET parameterized by the size of the desired solution is $W[2]$-complete on general graphs [7] and FPT on planar graphs [12].

Our results and techniques. In this article we provide the first explicit (and reasonably simple) polynomial-time data reduction rules for TOTAL DOMINATING SET on planar graphs, which lead

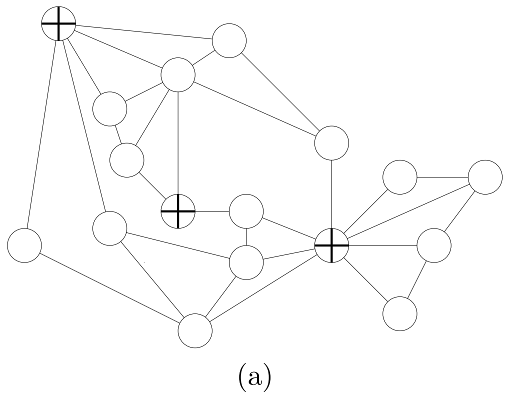


Figure 1: The vertices depicted with $\bigoplus$ define a minimal (a) dominating set; (b) TDS.

to a linear kernel for the problem. In particular, we prove the following theorem.

Theorem 1. *The* Total Dominating Set *problem parameterized by the solution size has a linear kernel on planar graphs. More precisely, there exists a polynomial-time algorithm that for each positive planar instance (G, k) returns an equivalence instance (G', k) such that $|V(G')| \leq 695 \cdot k$.*

This result complements several explicit linear kernels on planar graphs for other domination problems such as Dominating Set [1], Edge Dominating Set [15], Efficient Dominating Set [15], or Connected Dominating Set [14,24]. Our techniques are much inspired from those of Alber *et al.* [1] for Dominating Set. Namely, the rough idea of the method is to consider the neighborhood of each vertex and the neighborhood of each pair of vertices, and to identify some vertices that can be removed without changing the size of a smallest total dominating set. The corresponding reduction rules are called Rule 1 and Rule 2, respectively. Crucial to this approach is to decompose the input planar graph (which we assume to be already embedded in the plane; such a graph is called *plane*) into so-called *regions* which contain all vertices but $O(k)$ of them. Then it just remains to prove that in a reduced plane graph the total number of regions is $O(k)$ and that each of them contains $O(1)$ vertices, implying that the total number of vertices in the reduced instance is $O(k)$.

The main difference of our approach with respect to [1] lies in Rule 2. More precisely, due to the particularities of our problem, we need to distinguish more possibilities according to the neighborhood of a pair of vertices, which makes our rule slightly more involved. In particular, while in [1] the region decomposition is only used for the analysis of the reduced graph, we also use regions in order to reduce the input graph.

Organization of the paper. In Section 2 we provide two results independent from Theorem 1. Namely, we first give a simple proof of the NP-completeness of Total Dominating Set on planar graphs. This result was already claimed in [13,28] but we were not able to find the proofs. We then prove that Total Dominating Set satisfies the conditions in order to fall into the framework of Bodlaender *et al.* [3], and therefore it follows that there exists a linear kernel for this problem on planar graphs (and more generally, on graphs of bounded genus). In Section 4 we describe our reduction rules for Total Dominating Set when the input graph is embedded in the plane, and

in Section 5 we prove that the size of a reduced plane YES-instance is linear in the size of the desired total dominating set. In Section 6 we conclude with some directions for further research.

2 Preliminary results

In Subsection 2.1 we provide a simple proof of the NP-completeness of TOTAL DOMINATING SET on planar graphs, and in Subsection 2.2 we show that TOTAL DOMINATING SET satisfies the general conditions of the meta-theorem of Bodlaender *et al.* [3].

2.1 NP-completeness of total domination on planar graphs

Theorem 2. TOTAL DOMINATING SET *is NP-complete on planar graphs.*

Proof. Let $G = (V, E)$ be a planar graph and $D \subseteq V$. Checking whether D is a TDS can be clearly made in time $O(n^2)$, so TOTAL DOMINATING SET is in NP. We proceed to give a reduction from VERTEX COVER on planar graphs, which is known to be NP-hard [13].

Let $(G = (V, E), k)$ be an instance of VERTEX COVER, where G is planar. We construct an instance $(G' = (V', E'), k' = k + 2 \cdot |V|)$ of TOTAL DOMINATING SET, where each edge of G is replaced by a path with two edges, and a gadget of four vertices is added to each vertex of G. Fig. 2 shows the gadget that replaces an edge $\{v, w\}$. More precisely, $V' = \{v_1, v_2, v_3, v_4, v, e_{u,w} : v \in V, \{u, w\} \in E\} = \bigcup_{i \in [1,4]} V_i \cup V \cup V_E$, where $V_i = \bigcup_{v \in V} v_i$ and $V_E = \{e_{u,v} : \{u, v\} \in E\}$, and $E' = \{\{u, e_{u,v}\}, \{v, e_{u,v}\} : \{u, v\} \in E\} \cup \{\{v, v_1\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\} : v \in V\}$. Note that if G is planar, then G' is clearly planar as well.

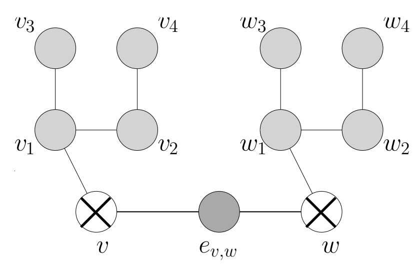


Figure 2: Gadget corresponding to an edge $\{v, w\}$. Vertices depicted with $\otimes$ correspond to vertices from V, while grey vertices are new.

Let D be a vertex cover in G of size k. We define a TDS of G' as $D' = D \cup V_1 \cup V_2$. We have that $|D'| = k + 2 \cdot |V|$. Vertices from V_E are dominated in G' since D covers all edges of G. Vertices from V,V_2, and V_3 are dominated by vertices from V_1, and vertices from V_1 and V_4 are dominated by vertices from V_2.

Conversely, let D' be a TDS in G' of size k'. We define $D = D' \cap V$. Edges from E are covered in G since vertices from V_E are dominated by vertices from D'. We have $|D| \leq k' - 2 \cdot |V|$, since V_1 and V_2 are necessarily included in D' (vertices from V_3 and V_4 need to be dominated by vertices from V_1 and V_2). Therefore, TOTAL DOMINATING SET is NP-complete on planar graphs. $\square$

2.2 Meta-kernelization for total dominating set

Before proving Lemma 1, we need some definitions taken from [3].

Given a graph $G = (V, E)$, a subset $S \subseteq V$, and an integer $r \geq 1$, $R^r_G(S)$ denotes the set of vertices whose radial distance from S is at most r in G. Let $\Pi \subseteq \mathcal{G}_g \times \mathbb{N}$ be a parameterized problem in graphs of genus g. We say that Π is *r-compact* (for $r \in \mathbb{N}$) if for all instances $(G = (V, E), k \in \mathbb{N}) \in \Pi$ there exists a set of vertices $S \subseteq V$ and an embedding of G such that $|S| \leq r \cdot k$ and $R^r_G(S) = V$. A problem Π is *compact* if there exists an r such that Π is r-compact. A graph $G = (V, E)$ is *t-boundaried* if it contains a set of t vertices, called $\delta(G)$, labeled from 1 to t. We call $\mathcal{B}_t$ the class of these graphs. The *gluing* of two t-boundaried graphs G_1 and G_2 is the operation $G_1 \oplus G_2$ which identifies vertices from $\delta(G_1)$ and $\delta(G_2)$ with the same label. Two labeled vertices are neighbors in $G_1 \oplus G_2$ if they are neighbors in G_1 or in G_2. Given t-boundaried graph $G = (V, E)$, we define the function $\zeta_G : G' = (V', E') \in \mathcal{B}_t, S' \subseteq V' \mapsto \min\{|S| : S \subseteq V \text{ and } P_\Pi(G \oplus G', S \cup S') \text{ is true }\}$. (If there is no such S, then ζ_G is undefined.) Assume that Π is a MSO minimization problem. We say that Π is *(strongly) monotone* if there exists a function f such that for any t-boundaried graph G there is a subset of vertices S such that for all (G', S') with $\zeta(G', S')$ defined, $P_\Pi(G \oplus G', S \cup S')$ is verified and $|S| \leq \zeta(G', S') + f(t)$.

The following theorem is an immediate consequence of [3, Theorem 2 and Lemma 12].

Theorem 3. *Let $\Pi \subseteq \mathcal{G}_g \times \mathbb{N}$ be a parameterized problem on graphs of genus g. If Π is monotone and Π or $\bar{\Pi}$ is compact, then Π admits a linear kernel.*

Lemma 1. *There exists a linear kernel for* Total Dominating Set *on graphs of bounded genus.*

Proof. By Theorem 3, we just need to prove that Total Dominating Set is monotone and compact. Let G be a t-boundaried graph and let $k \in \mathbb{N}$. By definition of Total Dominating Set, if (G, k) admits a solution D then $|D| \leq k$ and $R^1_G(D) = V$. So Total Dominating Set is clearly compact (for $r = 1$). If D is a solution of G, then D and G verify the MSO predicate $P(G, D) = \{\forall v \in V, \exists d \in D, adj(v, d)\}$. Let $D^{(2)}$ be a TDS of minimal size for G and $D^{(3)} \subseteq N(\delta(G))$ containing a neighbor in G for each vertex from $\delta(G)$, if such a neighbor exists. We define $D := D^{(2)} \cup D^{(3)} \cup \delta(G)$. Let (G', D') with $G' = (V', E')$ be a t-boundaried graph, $D' \subseteq V'$, and $\zeta_G(G', D')$ defined. We have that $D \cup D'$ is a TDS for $G \oplus G'$ and that $|D| \leq \zeta(G', D') + 2 \cdot t$. Notice that if $v \in \delta(G)$ is isolated in G (so, without neighbor from $D^{(3)}$) then it has necessary a neighbor in D' (otherwise $\zeta(G', D')$ is undefined). Thus, Total Dominating Set is monotone. $\square$

3 Definitions

In this section we give necessary definitions for our reduction rules and the analysis of the kernel size. Most of them are defined by Alber *et al.* in [1] for obtaining a linear kernel for Dominating Set on planar graphs. Given a graph G, we denote by $\gamma_t(G)$ the size of a smallest TDS of G. We first split the neighborhood of a vertex and a pair of vertices into three subsets.

Definition 1. *Let $G = (V, E)$ be a graph and let $v \in V$. We note $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v. We split $N(v)$ into three subsets:*

$$N_1(v) = \{u \in N(v) : N(u) \setminus (N(v) \cup \{v\}) \neq \emptyset\},$$

$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\},$$

$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)).$$

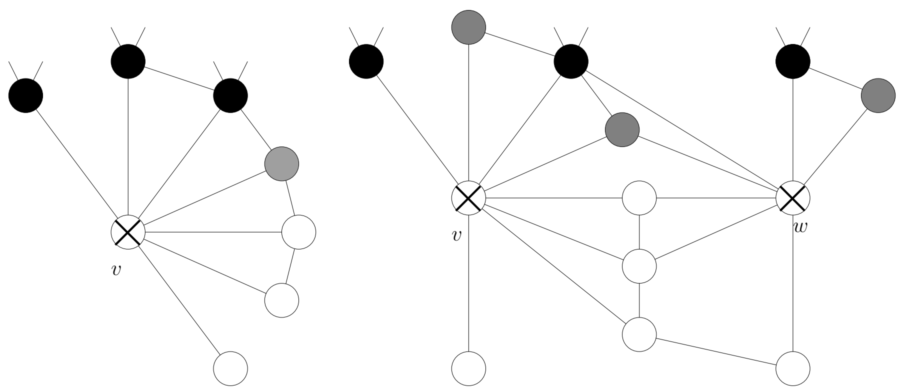
Let $v, w \in V$ be two distinct vertices. We note $N(v,w) = N(v) \cup N(w) \setminus \{v,w\}$ the neighborhood of the pair $\{v,w\}$. Similarly, we split $N(v,w)$ into three subsets:

$$N_1(v,w) = \{u \in N(v,w) : N(u) \setminus (N(v,w) \cup \{v,w\}) \neq \emptyset\},$$

$$N_2(v,w) = \{u \in N(v,w) \setminus N_1(v,w) : N(u) \cap N_1(v) \neq \emptyset\},$$

$$N_3(v,w) = N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)).$$

For $i,j \in [1,3]$ with $i \neq j$ and $v,w \in V$ with $v \neq w$, we define $N_{i,j}(v) = N_i(v) \cup N_j(v)$ and $N_{i,j}(v,w) = N_i(v,w) \cup N_j(v,w)$. Fig. 3 gives an example of the neighborhood of a vertex v and a pair of vertices $\{v,w\}$.



(a) (b)

Figure 3: Neighborhood of (a) a vertex v; and (b) a pair (v,w). Considered vertices are depicted with $\bigotimes$, $N_1(v)$ and $N_1(v,w)$ are in black, $N_2(v)$ and $N_2(v,w)$ are in grey, and $N_3(v)$ and $N_3(v,w)$ are in white.

In order to bound the kernel size, the concept of *region* will play a fundamental role.

Definition 2. *Let $G = (V,E)$ be a plane graph and let $v,w \in V$ be two distinct vertices. A* region *$R(v,w)$ between v,w is a closed subset of the plane such that:*

- *the boundary of $R(v,w)$ is formed by two simple paths connecting v and w, and the length of each path (that is, its number of edges) is at most three, and*
- *all vertices strictly inside $R(v,w)$ (that is, outside the boundary) are from $N(v,w)$.*

We note $V(R(v,w))$ the set of vertices inside $R(v,w)$ (that is, vertices strictly inside, on the boundary, and the two extremities v,w), and we say that $|V(R(v,w))|$ is the region size. *A* simple region *$R(v,w)$ is a region such that all vertices of $V(R(v,w))$ except for v,w are common neighbors of v and w, i.e., $V(R(v,w)) \setminus \{v,w\} \subseteq N(v) \cap N(w)$. A subset of vertices S is* covered *by a set of regions $\mathcal{R}$ if $S \subseteq \bigcup_{R(v,w) \in \mathcal{R}} V(R(v,w))$. Fig. 4 gives an example of region and simple region.*

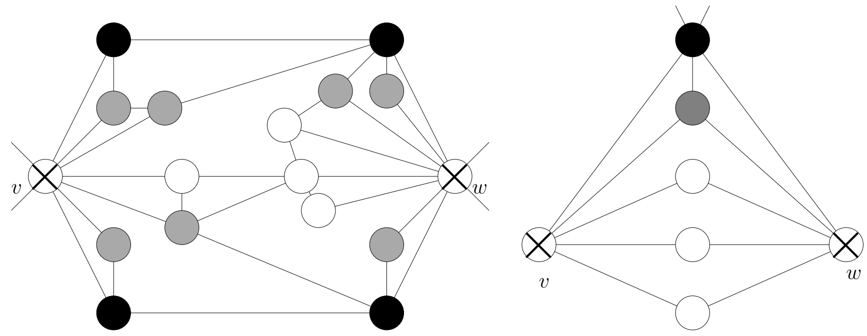


Figure 4: (a) A region; and (b) a simple region. Vertices from $N_1(v, w)$ are necessarily on the boundary, and v and w have potentially other neighbors.

Definition 3. *Let $G = (V, E)$ be a plane graph and let $S \subseteq V$. An S-*decomposition *of G is set of regions $\mathcal{R}$ between pairs of vertices in S such that:*

- *for $R(v, w) \in \mathcal{R}$ no vertex from S (except for v, w) lies in $V(R(v, w))$, and*
- *any two regions $R_1(v_1, w_1), R_2(v_2, w_2) \in \mathcal{R}$ have only vertices of the boundary in common.*

We note $V(\mathcal{R}) = \bigcup_{R(v,w)\in\mathcal{R}} V(R(v, w))$. An S-decomposition is maximal *if there is no region $R(v, w) \notin \mathcal{R}$ such that $\mathcal{R}' = \mathcal{R} \cup \{R(v, w)\}$ is an S-decomposition with $V(\mathcal{R}) \subsetneq V(\mathcal{R}')$. Fig. 5 gives an example of maximal S-decomposition, where S a dominating set.*

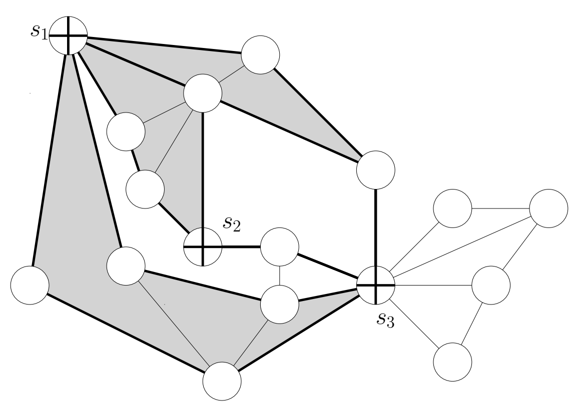


Figure 5: A maximal S-decomposition, where $S = \{s_1, s_2, s_3\}$ is the set of $\bigoplus$ vertices. There are two regions between s_1 and s_3, one between s_1 and s_2, and 1 between s_2 and s_3. Some neighbors of s_3 do not belong to any region.

We also define the notion of total domination for a subset of vertices.

Definition 4. *Let $G = (V, E)$ be a graph and let $S \subseteq V$. An S-*TDS *$D_S \subseteq N(S) \cup S$ is a set which dominates S such that the induced graph $G[D_S]$ contains no isolated vertex from S (but may contain isolated vertices from $N(S)$).*

4 Reduction rules

In this section we give reduction rules for Total Dominating Set on planar graphs. For each vertex and each pair of vertices, we consider the three neighborhoods defined above, and we show that in some cases the third one ($N_3(v)$ or $N_3(v,w)$) can be reduced to a smaller one. These rules are similar to those of [1] for Dominating Set, but the second one is slightly more involved.

4.1 Reduction for a vertex

Notice that vertices from $N_3(v)$ cannot be dominated by a vertex from $N_1(v)$. Even if a vertex $u \in N_{2,3}(v)$ dominates all the vertices in $N_3(v)$, it is clear that v is an equal or better choice, since v dominates the three subsets of $N(v)$ and in addition any neighbor of u is also a neighbor of v, so any vertex that dominates u also dominates v as well (recall that all vertices need to be dominated in Total Dominating Set). This discussion motivates the first reduction rule.

Rule 1: Let $G = (V, E)$ be a plane graph. If there is a vertex $v \in V$ such that $|N_3(v)| \geq 1$:

- remove $N_{2,3}(v)$ from G, and
- add a new vertex v' with the edge $\{v, v'\}$.

The new vertex v' enforces to include v in the TDS, and can also be used to dominate v. Fig. 6 gives an example of the application of Rule 1 on a vertex v.

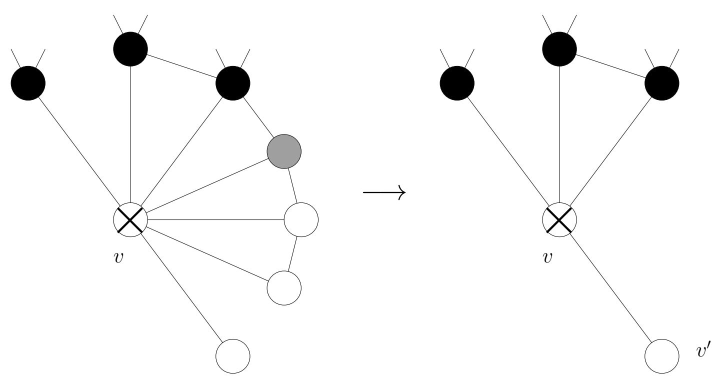


Figure 6: Example of Rule 1 applied on a vertex v. The original white and grey vertices are removed, and a new vertex v' is added.

Lemma 2. *Let $G = (V, E)$ be a plane graph, let $v \in V$ and let $G' = (V', E')$ be the graph obtained by the application of Rule 1 on v. Then $\gamma_t(G) = \gamma_t(G')$.*

Proof. Let first D be a TDS of G. We define a TDS D' of G' as follows: if $D \cap N_{2,3}(v) \neq \emptyset$ we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$, and $D' = D$ otherwise. Recall that by assumption $N_3(v) \neq \emptyset$. If $v \notin D$, then there exists $u \in N_{2,3}(v) \cap D$, and since $N(u) \subseteq N(v)$ we can exchange in D u for v. Now we can assume that $v \in D$. If Rule 1 removes a vertex u from D, the neighbors of u are dominated

since they are also neighbors of v. Moreover, v' is dominated by v, and v is dominated by v' added to D'. If the reduction removes no vertex from D, D' is clearly a TDS, since v is dominated by a vertex from $N_1(v)$. Therefore, as $|D'| \leq |D|$ we have that $\gamma_t(G) \geq \gamma_t(G')$.

Let now D' be a TDS of G'. We define a TDS D of G as follows: if $v' \in D'$ we set $D = D' \setminus \{v'\} \cup \{u\}$ for any vertex $u \in N_3(v)$, and $D = D'$ otherwise. Since v' needs to be dominated, necessarily $v \in D'$. Rule 1 may remove only neighbors of v, so they are dominated by v in G. If we add u to D then v is dominated by u, otherwise v is clearly dominated by a vertex from $N_1(v)$. Therefore, as $|D| \leq |D'|$ we have that $\gamma_t(G) \leq \gamma_t(G')$. □

As Rule 1 is the same as the one applied in [1] for Dominating Set, the next lemma follows directly from [1, Lemma 2].

Lemma 3. *Given a plane graph G of n vertices, G is reduced under Rule 1 in time $O(n)$.*

4.2 Reduction for a pair of vertices

The reduction for a pair of vertices is based on the same idea than the previous one. Nevertheless the choices here may be more involved: indeed, two such vertices v, w are not always the best choice, as it could be better to choose two or three vertices from $N_{2,3}(v, w)$ which dominate themselves rather than v, w, which may not be adjacent and need new vertices in order to be dominated.

We consider the $N_3(v, w)$-TDS of size at most three containing either v or w. Given v, w, we note $\mathcal{D}_v = \{D_{N_3} \subseteq N_{2,3}(v, w) \cup \{v\} : N_3(v, w) \subseteq \bigcup_{u \in D_{N_3}} N(u), |D_{N_3}| \leq 3, v \in D_{N_3}\}$, and $\mathcal{D}_w$ is defined symmetrically. We note that we could restrict the definition to minimal $N_3(v, w)$-TDS.

In Rule 2 we need to construct simple regions between given pairs of vertices. The following procedure provides a simple way to identify maximal simple region between two vertices. Since we assume that the input graph is embedded in the plane, the neighbors of a vertex are circularly ordered.

How to compute maximal simple regions: Let $G = (V, E)$ be a plane graph and let $u, u' \in V$.

- start with a non-common neighbor of u such that the next one, namely v, is a common neighbor of u and u',
- create a new simple region containing v,
- add to this region all consecutive common neighbors which follow v in the ordering,
- skip non-common neighbors,
- repeat the above procedure until all neighbors are treated.

We are now ready to state our second reduction rule.

Rule 2: Let $G = (V, E)$ be a plane graph. If there exist two distinct vertices $v, w \in V$ such that there is no $N_3(v, w)$-TDS of size at most 3 in $N_{2,3}(v, w)$:

1. if $\mathcal{D}_v = \emptyset$ and $\mathcal{D}_w = \emptyset$,
   - remove $N_3(v, w)$ from G,
   - add a new vertex v' with the edge $\{v, v'\}$,
   - add a new vertex w' with the edge $\{w, w'\}$,
   - if there exists a common neighbor of v and w in $N_3(v, w)$, add a new vertex y with the edges $\{v, y\}$ and $\{w, y\}$;
2. if $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$,
   - take arbitrarily a $N_3(v, w)$-TDS $D_{N_3} \in \mathcal{D}_v$,
   - for all $u \in D_{N_3}$,
     make maximal simple regions between w and u using the procedure described above,
   - while there is a simple region of size more than 7,
     remove a vertex strictly inside that region,
   - do symmetrically for $\mathcal{D}_w$;
3. if $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w = \emptyset$,
   - add a new vertex v' with the edge $\{v, v'\}$,
   - take arbitrarily a $N_3(v, w)$-TDS $D_{N_3} \in \mathcal{D}_v$,
   - for all $u \in D_{N_3}$, $u \neq v$,
     make maximal simple regions between w and u using the procedure described above,
   - while there is a simple region of size more than 7,
     remove a vertex strictly inside that region,
   - remove $N_{2,3}(v) \cap N_3(v, w)$ from G,
   - for all vertices u on the boundary of the regions constructed above,
     make maximal simple regions between v and u using the procedure described above,
   - while there is a simple region of size more than 7,
     remove a vertex strictly inside that region;
4. if $\mathcal{D}_v = \emptyset$ and $\mathcal{D}_w \neq \emptyset$,
   - symmetrically to Case 3.

Let us point out that in our Rule 2 we use the concept of region, whereas Alber *et al.* [1] use it only for the analysis of the kernel size. Note also that in Rule 2 we do not impose simple regions to be disjoint (whereas in the analysis we will need to consider only disjoint regions). Fig. 7 gives an example of the application of Rule 2 on a pair of vertices $\{v, w\}$.

In the following facts, we always use G to denote a plane graph on which Rule 2 can be applied and G' to denote the resulting plane graph on which Rule 2 has been applied.

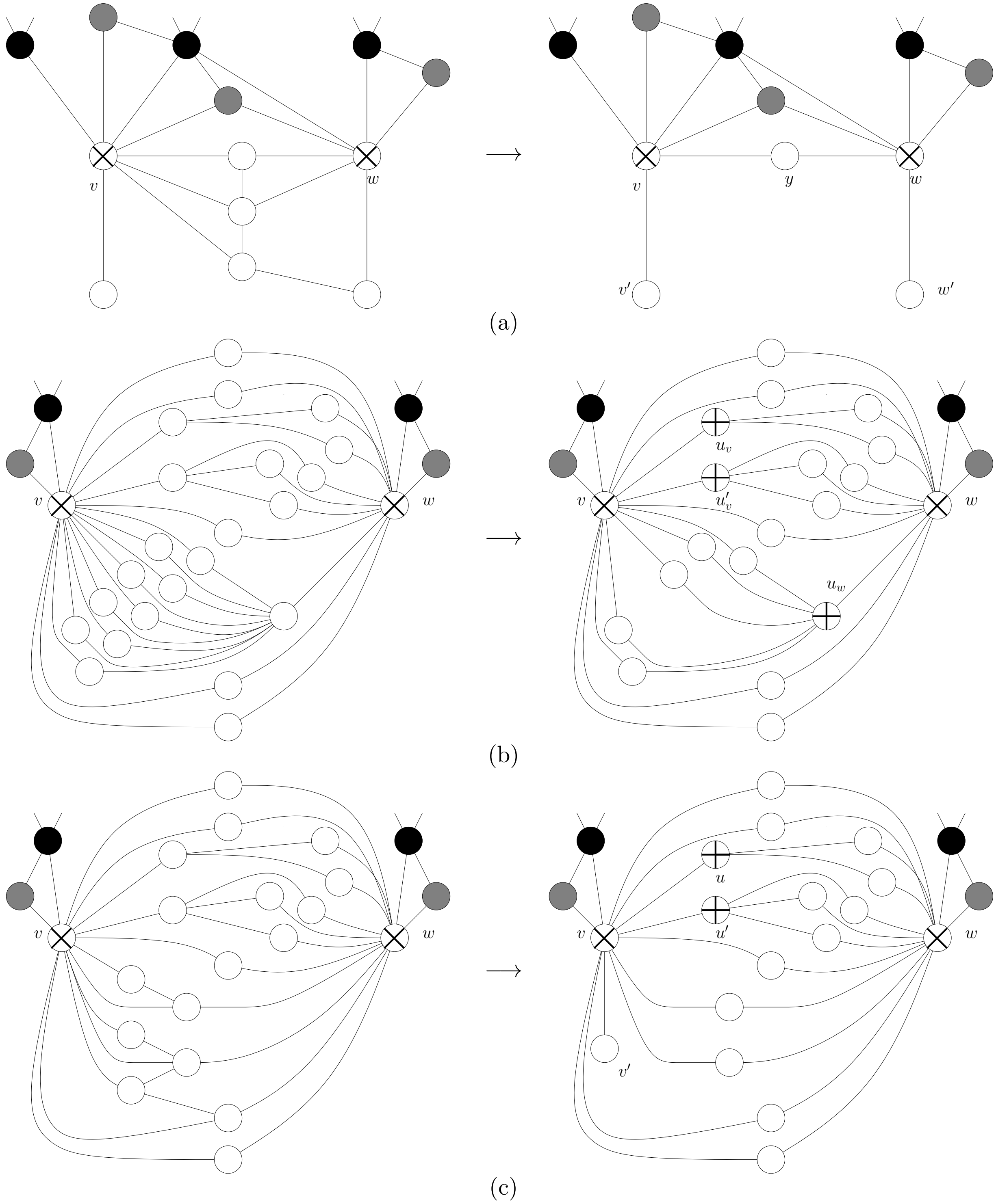


Figure 7: Example of Rule 2 applied on a pair $\{v, w\}$: (a) in Case 1, $N_3(v, w)$ is replaced by v', w', y ; (b) in Case 2, two vertices u_v, u'_v together with v and one vertex u_w together with w form an $N_3(v, w)$-TDS, and in this case the region between u_w and v is reduced ; (c) in Case 3, two vertices u, u' together with v form an $N_3(v, w)$-TDS, and in this case the neighborhood of v is reduced.

Fact 1. *If Rule 2 is applied on $\{v, w\}$, we can assume that v or w belong to any TDS of G.*

Proof. According to the initial condition for Rule 2 to be applied, any TDS of G must contain either a vertex from $\{v, w\}$, or (at least) 4 vertices from $N_{2,3}(v, w)$. In the latter case these 4 vertices can be replaced by v, w, s, t where s, t are neighbors of v, w respectively. So we can always assume that a TDS contains v or w. □

Fact 2. *If $\mathcal{D}_w = \emptyset$ (resp. $\mathcal{D}_v = \emptyset$), then we can assume that v (resp. w) belongs to any TDS of G.*

Proof. Assume w.l.o.g. that $\mathcal{D}_w = \emptyset$. Then no set of the form $\{w\}$, $\{w, u\}$, or $\{w, u, u'\}$ (with $u, u' \in N_{2,3}(v, w)$) can dominate $N_3(v, w)$. So we need at least 4 vertices to dominate $N_3(v, w)$ without v, and as in the proof of Fact 1 we can replace them by v, w, s, t. □

Fact 3. *If $R(u, u')$ is a simple region in G (resp. G') of size more than 7, then we can assume that u or u' belong to any TDS of G. We can also assume that such a TDS does not contain vertices strictly inside $R(u, u')$.*

Proof. W.l.o.g., if D contains u and a vertex strictly inside $R(u, u')$, then this vertex can be replaced by a vertex on the boundary. Otherwise, since $R(u, u')$ is simple, D contains at least 2 vertices from $V(R(u, u'))$, with one of them strictly inside, which can be replaced by u. □

Fig. 8 gives an example of a simple region with a dominating vertex strictly inside.

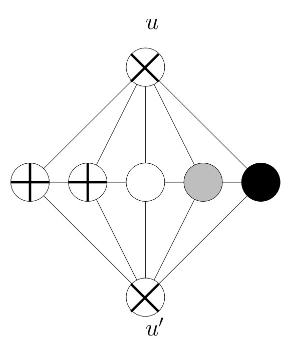


Figure 8: A simple region dominated by two $\bigoplus$ vertices and 1 black vertex (from $N_1(v, w)$). Vertex u can replace the $\bigoplus$ vertex which is strictly inside the region.

Fact 4. *If Rule 2 has been applied on $\{v, w\}$ and a neighbor v' (resp. w') of v (resp. w) has been added, then v belongs to any TDS of G', and v has a neighbor in $N_3(v, w)$ that is not neighbor of w (resp. v), i.e., $(N(v) \setminus N(w)) \cap N_3(v, w) \neq \emptyset$.*

Proof. W.l.o.g, if v' is added, then it can only be dominated by v, his only neighbor. Moreover v' is added only if $\mathcal{D}_w = \emptyset$, in particular w does not dominate $N_3(v, w)$ (as in the proof of Fact 2), so v has a neighbor which is not adjacent to w. □

Lemma 4. *Let $G = (V, E)$ be a plane graph, let $v, w \in V$, and let $G' = (V', E')$ be the graph obtained by the application of Rule 2 on the pair $\{v, w\}$. Then $\gamma_t(G) = \gamma_t(G')$.*

Proof. We prove independently each case of Rule 2, and we make all assumptions guaranteed by the previous facts. In what follows, D (resp. D') will denote a TDS of G (resp. G'). Note that when applying Rule 2 on a pair of vertices $\{v, w\}$, we only have to care about the neighbors of $\{v, w\}$, as in the rest of the graph the TDS will not be affected.

1. Given D, we define D' as follows: we start with $D' = D \setminus N_3(v, w)$ and we add to D' vertices y, v', or w' according to whether Rule 2 has removed some vertex in $D \cap N(v) \cap N(w)$, $D \cap N(v)$, or $D \cap N(w)$, respectively. Note that in all cases it holds that $|D'| \leq |D|$.

   Let us now prove that D' is a TDS of G'. Because of Fact 2, we have that $v, w \in D$. If Rule 2 removes some vertex u from D, as the neighbors of u are also neighbors of v or w, they are dominated by v or w. Moreover v', w', y are dominated by v, w. And v, w are dominated by v', w', or y added to D'. If Rule 2 removes no vertex from D, D' is clearly a dominating set, since v, w are dominated by vertices from $N_{1,2}(v, w)$. Therefore we have $\gamma_t(G) \geq \gamma_t(G')$.

   Conversely, given D' we define D as follows: we start with $D = D' \setminus \{y, v', w'\}$ and we add to D vertices $u_{vw} \in N(v) \cap N(w) \cap N_3(v, w)$, $u_v \in N_3(v, w) \setminus N(w)$, or $u_w \in N_3(v, w) \setminus N(v)$, according to whether vertex y, v', or w' belongs to D', respectively (the added u_{vw} exists by definition and u_v, u_w exist by Fact 4). Note that in all cases it holds that $|D| \leq |D'|$.

   Let us now prove that D is a TDS of G. Because of vertices v', w', we have that $v, w \in D'$, and therefore also to D. Rule 2 removes only neighbors of v or w, so they are dominated by v or w. If we add u_{vw} then v, w is dominated by u_{vw}, if we add u_v then v is dominated by u_v, if we add u_w then w is dominated by u_w, and otherwise v, w are clearly dominated by a vertex from $N_{1,2}(v, w)$. Therefore we have that $\gamma_t(G) \leq \gamma_t(G')$.

2. In this case we just take $D' = D$. Let us now prove that D' is indeed a TDS of G'. Rule 2 removes only vertices in simple regions of size more than 7, and by Fact 3 they do not belong to D. Therefore we have $\gamma_t(G) \geq \gamma_t(G')$.

   Conversely, we also take $D = D'$. Let us now prove that D is a TDS of G. By Fact 3, simple regions $R(u, u')$ of size at least 7 are dominated by u or u'. Rule 2 removes only vertices strictly inside these regions, so they are dominated by u or u'. Therefore we have that $\gamma_t(G) \leq \gamma_t(G')$.

3. We define D' as follows: we start with $D' = D \setminus N_3(v, w)$ and we add to D' vertex v' according to whether Rule 2 has removed some vertex in $D \cap N(v)$ (by Fact 3, vertices removed in simple regions do not belong to D).

   Let us now prove that D' is a TDS of G'. By Fact 2 we have that $v \in D$. If Rule 2 removes a vertex u from D, as the neighbors of u are also neighbors of v, they are dominated by v. Moreover v' is dominated by v, and v is dominated by v' added to D'. If Rule 2 removes no vertex from D, D' is clearly a dominating set, since v is dominated by vertices from $N_{1,2}(v, w)$. Therefore we have $\gamma_t(G) \geq \gamma_t(G')$.

   Conversely, we define D as follows: we start with $D = D' \setminus \{v'\}$ and we add to D a vertex $u_v \in N_3(v, w) \setminus N(w)$ according to whether vertex v' belongs to D' (note that such a vertex u_v exists by Fact 4).

   Let us now prove that D is a TDS of G. Because of vertex v' we have that $v \in D'$. Rule 2 removes neighbors of v, which are dominated by v, or vertices of simple regions of size more

than 7, which are dominated by Fact 3. If vertex u_v is added then it dominates v, and otherwise v was already dominated by vertices from $N_{1,2}(v,w)$. Therefore we have that $\gamma_t(G) \leq \gamma_t(G')$.

4. Symmetrically to Case 3.

$\square$

Lemma 5. *A plane graph G can be reduced under Rule 2 in time $O(n^6)$.*

Proof. $N(v,w)$ can be split into $N_i(v,w)$ for $i \in [1,3]$ in time $O(n^2)$. Subsets of size at most 3 are tested to be dominating in time $\binom{n}{3} \cdot n = O(n^4)$. Vertices and edges are added or removed in time $O(n^2)$. Simple regions are made and reduced in time $O(n^2)$. Rule 2 is potentially applied on each pair of vertices, so at most $\binom{n}{2} = O(n^2)$ times. So the overall complexity is $O(n^6)$. $\square$

5 Bounding the size of the kernel

In this section we show that a plane graph reduced under Rules 1 and 2 (that is, a plane graph for which Rules 1 or 2 cannot be applied anymore) has linear size in $|D|$, the size of a TDS. We show that, given a solution D of size k, there exist a maximal region decomposition $\mathcal{R}$ such that:

- $\mathcal{R}$ has a $O(k)$ number of regions,
- $\mathcal{R}$ covers all vertices but $O(k)$ of them, and
- each region of $\mathcal{R}$ has $O(1)$ vertices.

The three following propositions treat respectively each of the above statements. We first need a lemma to bound the size of simple regions.

Lemma 6. *Given a plane graph $G = (V, E)$ reduced under Rules 1 and 2, a simple region $R(v,w)$ in G has size at most 11, that is, it contains at most 9 vertices different from $\{v,w\}$.*

Proof. We bound separately the vertices of $R(v,w)$ which are from $N_1(v,w)$, $N_2(v,w)$, and $N_3(v,w)$. The vertices in $N_1(v,w)$ need to be on the boundary of the region $R(v,w)$, and since it is simple, they are at most 2. The vertices of $N_2(v,w)$ need to be neighbors of some vertex from $N_1(v,w)$. Since $R(v,w)$ is simple and by planarity, each vertex of $N_1(v,w)$ has at most one neighbor in $N_2(v,w)$. Since G is reduced, Rule 2 has eventually considered the pair $\{v,w\}$, so $N_3(v,w)$ can take three forms. Indeed, if Rule 2 has not been applied on $\{v,w\}$, then there is an $N_3(v,w)$-TDS (containing neither v nor w) of size at most 3, and in particular this set dominates the vertices strictly inside $R(v,w)$. By planarity, we have at most 5 vertices strictly inside. Fig. 9 gives an example of a worst possible case when Rule 2 has not been applied. If Case 1 of Rule 2 has been applied, then $N_3(v,w)$ is completely removed, and only the new vertex y can be in $R(v,w)$. Finally, if Cases 2, 3, or 4 of Rule 2 have been applied, we can directly bound the total size of the simple region. Indeed, in this case $R(v,w)$ is included in a simple region between v,w constructed by the procedure to compute maximal simple regions (otherwise it would contradict the maximality of the regions), which is reduced and therefore has size at most 7. Summarizing we have $|V(R(v,w))| \leq \max(2+4+5, 2+4+1, 7) = 11$. $\square$

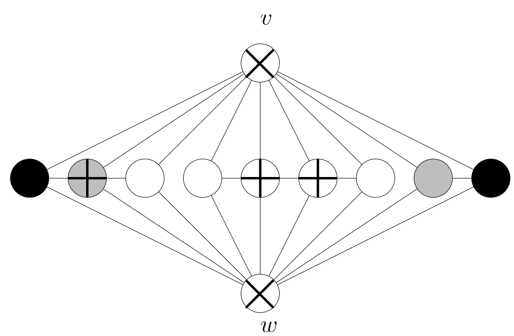


Figure 9: The above simple region contains at most two black vertices, two grey ones, and at most five white ones can be dominated by $\bigoplus$ vertices.

The following proposition corresponds to [1, Proposition 1 and Lemma 5]. It can be safely applied to a graph reduced under our rules, as it holds for any dominating set (note that a total dominating set is in particular a dominating set), and the authors of [1] do not use specific properties supplied by their reductions for constructing such a maximal D-decomposition.

Proposition 1. *Let $G = (V, E)$ be a plane graph and let D be a TDS of G. Then there exists a maximal D-decomposition $\mathcal{R}$ such that $|\mathcal{R}| \leq 3 \cdot |D| - 6$.*

We can now bound the number of vertices that do not belong to any region.

Proposition 2. *Let $G = (V, E)$ be a plane graph reduced under Rule 1 and 2, and let D be a TDS of G. If $\mathcal{R}$ is a maximal D-decomposition, then $|V \setminus V(\mathcal{R})| \leq 218 \cdot |D|$.*

Proof. We follow the scheme of the proof provided by Alber *et al.* [1, Proposition 2]. The crucial point is that this proof does not depend on Rule 2, and therefore we have just to change the bound on simple regions size provided by our Lemma 6. In fact, we prove that $|V \setminus V(\mathcal{R})| \leq 72 \cdot |\mathcal{R}| + 2 \cdot |D|$, and then Proposition 1 provides the linear bound in $|D|$.

Let $\mathcal{R}$ be a maximal D-decomposition and let $v \in D$. We bound separately the vertices of $V \setminus V(\mathcal{R})$ which are from $N_1(v, w)$, $N_2(v, w)$, or $N_3(v, w)$. First, we have that $N_1(v) \subseteq V(\mathcal{R})$ [1, Lemma 6]. Hence $\bigcup_{v \in D} N_1(v) \setminus V(\mathcal{R}) = \emptyset$. Then, $N_2(v) \setminus V(\mathcal{R})$ can be covered by 4ℓ [1, Proposition 2] simple regions between v and some vertices from $N_1(v)$ on boundaries of $\mathcal{R}$, where ℓ is the number of regions in $\mathcal{R}$ adjacent to v, i.e., $\ell = |\{R(v, w) \in \mathcal{R}, w \in D\}|$. By Lemma 6 we can bound the size of these simple regions and hence, making the sum for each vertex, $|\bigcup_{v \in D} N_2(v) \setminus V(\mathcal{R})| \leq 9 \cdot 4 \cdot 2 \cdot |\mathcal{R}|$. Finally, we have $|N_3(v)| \leq 1$, since G is reduced under Rule 1. Hence $|\bigcup_{v \in D} N_3(v) \setminus V(\mathcal{R})| \leq |D|$. Taking the union of neighborhoods and adding the set D we obtain $|V \setminus V(\mathcal{R})| \leq 0 + 72 \cdot |\mathcal{R}| + |D| + |D|$. □

Fact 5. *Given a plane graph $G = (V, E)$ and a region $R(v, w)$, $|N_1(v, w) \cap V(R)| \leq 4$ and $N_2(v, w) \cap V(R)$ can be covered by 6 simple regions.*

Proof. Let v, u_1, u_2, w and v, u_3, u_4, w be the vertices that define the two paths of the boundary of $R(v, w)$ (clearly if the paths are smaller then we obtain a smaller bound). By definition of $N_1(v, w)$,

vertices from $N_1(v,w) \cap V(R)$ are on the boundary of R. Hence $|N_1(v,w) \cap V(R)| \leq 4$. By definition of $N_2(v,w)$, vertices from $N_2(v,w) \cap V(R)$ are common neighbors of v or w and u_i ($i \in [1,4]$). By planarity, $N_2(v,w) \cap V(R)$ can be covered by 6 simple regions (among 8 pairs of vertices). Fig. 10 gives an example with 3 simple regions for v and 3 simple regions for w. □

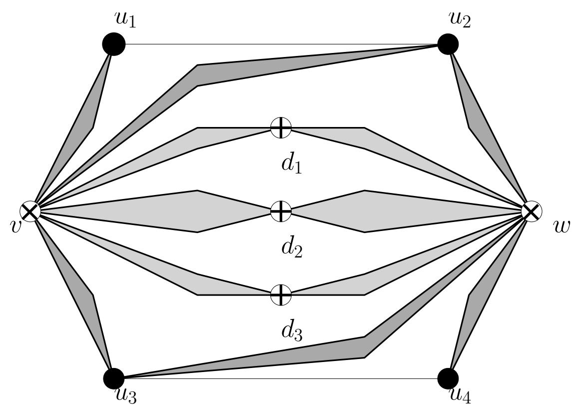


Figure 10: A region covered by simple regions when Rule 2 has not been applied. $N_2(v,w)$ is covered by 6 dark grey regions and $N_3(v,w)$ is covered by 6 light grey regions.

Fact 6. *Given a plane graph $G = (V,E)$ reduced under Rule 2 and a region $R(v,w)$, if $\mathcal{D}_v \neq \emptyset$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v,w) \cap R(v,w)$ can be covered by: 11 simple regions if $\mathcal{D}_w \neq \emptyset$, or 14 simple regions if $N_{2,3}(v) \cap N_3(v,w) = \emptyset$.*

Notice that the condition of the first case (namely, that $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$) means that Case 2 of Rule 2 has been applied. We say that a vertex u is *pendant* from a vertex v if $N(u) = \{v\}$. If $N_3(v)$ contains only pendant vertices from v (v' for instance), then it is clear that we can choose an embedding (moving only these pendant neighbors) such that Fact 6 is also true. So the second case of Fact 6 can be applied when Cases 3 or 4 of Rule 2 have been applied.

Proof. In this proof we only consider vertices form $R(v,w)$, as we always consider sets intersecting $V(R(v,w))$. We split $N_3(v,w)$ into $N_3(v,w) \setminus N(w)$, $N_3(v,w) \setminus N(v)$, and $N_3(v,w) \cap N(v) \cap N(w)$.

Let $D_{N_3} = \{v,u,u'\} \in \mathcal{D}_v$ (it will become clear from the proof that if the set contains fewer vertices, then we obtain a smaller bound). Since D_{N_3} dominates $N_3(v,w)$, we have that vertices from $N_3(v,w) \setminus N(v)$ are common neighbors of w and either u or u'. By planarity, we can cover $N_3(v,w) \setminus N(v)$ with at most 3 simple regions between w and either u or u'. Fig. 11(a) gives a scheme of the worst possible case.

Now, assume that $\mathcal{D}_w \neq 0$. We can cover $N_3(v,w) \setminus N(w)$ by at most 3 simple regions (as in the previous case). There remain vertices of $N_3(v,w) \cap N(v) \cap N(w)$ which are common neighbors of v and w. By planarity, we can cover $N_3(v,w) \cap N(v) \cap N(w)$ with at most 5 simple regions between v and w. Fig. 11(b) gives a scheme of the worst possible case. Hence $N_3(v,w) \cap R(v,w)$ can be covered by at most $3+3+5 = 11$ simple regions.

Now, assume that $N_{2,3}(v) \cap N_3(v,w) = \emptyset$, that is, all neighbors of v in $N_3(v,w)$ are from $N_1(v)$. These vertices have neighbors from $N_{2,3}(v,w)$ (by definition of $N_3(v,w)$) and different from $N(v)$

(by definition of $N_1(v)$), that is, vertices from $N(w) \cap N_{2,3}(v,w)$. But we have already covered vertices from $N(w) \cap N_3(v,w)$ by simple regions and by Fact 5, vertices from $N(w) \cap N_2(v,w)$ are also covered by 4 simple regions. So vertices from $N_3(v,w) \setminus N(w)$ are common neighbors of v and a vertex on the boundary of previously considered simple regions. By planarity, only 8 vertices on the boundaries can be neighbors of the considered vertices (that is, the vertices in $N(v) \cap N_3(v,w)$), and therefore we can cover $N_3(v,w) \setminus N(w)$ by at most 8 simple regions between v and vertices on the boundaries. There remain vertices of $N_3(v,w) \cap N(v) \cap N(w)$, which are common neighbors of v and w. By planarity, we can cover $N_3(v,w) \cap N(v) \cap N(w)$ with at most 3 simple regions between v and w. Fig. 11(c) gives a scheme of the worst case. Hence $N_3(v,w) \cap R(v,w)$ can be covered by at most $3 + 8 + 3 = 14$ simple regions. $\square$

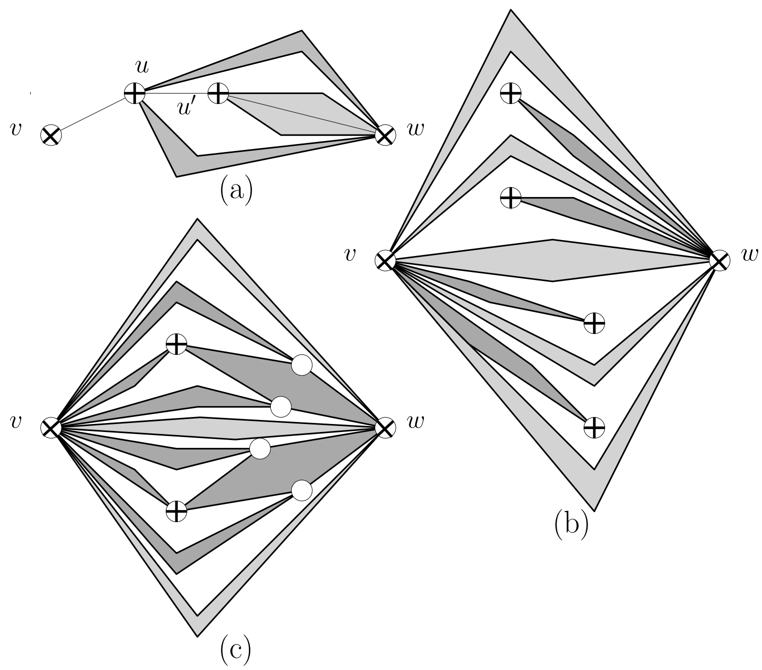


Figure 11: Covering of (a) the non-neighbors of v; (b) the common neighbors of v and w; (c) the non-neighbors of w in $N_3(v,w)$.

Note that the worst possible configuration for the first subset of $N_{2,3}(v,w)$ is not compatible with the worst configurations for the other subsets, so better bounds could be obtained with a more careful analysis. For the sake of simplicity, we provide this shorter proof here.

Proposition 3. *Let $G = (V,E)$ be a plane graph reduced under Rules 1 and 2, let D be a TDS of G, and let $v, w \in D$. Then a region R between v and w has size at most 159.*

Proof. By Fact 5 and using Lemma 6 to bound the size of simple regions, we have that $|N_1(v,w) \cap V(R)| \leq 4$ and $|N_2(v,w) \cap V(R)| \leq 6 \cdot 9 = 54$[1].

[1] It is worth mentioning that using the definition of *type* of simple region given in [1, Lemma 7] one could obtain a better bound, namely $|N_2(v,w) \cap V(R)| \leq 46$.

We now bound $|N_3(v,w) \cap V(R)|$ according to the application of Rule 2 on $\{v,w\}$. First, if Rule 2 is not applied, then $N_3(v,w)$ is dominated by at most three vertices d_1, d_2, d_3, and vertices from $N_3(v,w) \cap V(R)$ are common neighbors of either v or w and d_i ($i \in [1,3]$). By planarity, we can cover $N_3(v,w) \cap V(R)$ with at most 6 simple regions. Fig. 10 gives an example where all dominating vertices are in the considered region. Using Lemma 6 we get in this case $|N_3(v,w) \cap V(R)| \leq 6 \cdot 9 + 3 = 57$. If Case 1 of Rule 2 has been applied, then $N_3(v,w)$ consists of the newly added vertices, so in this case $|N_3(v,w) \cap V(R)| \leq 3$. If Case 2 of Rule 2 has been applied, then $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$. By Fact 6, we can cover $N_3(v,w) \cap V(R)$ with at most 11 simple regions, which are reduced by Rule 2. Hence $|N_3(v,w) \cap V(R)| \leq 11 \cdot 7 = 77$. Finally, if Cases 3 or 4 of Rule 2 have been applied, assume w.l.o.g. that $N_{2,3}(v) \cap N_3(v,w)$ consists of the newly added vertex v'. By Fact 6, we can cover $(N_3(v,w) \setminus \{v'\}) \cap V(R)$ with 14 simple regions, which are reduced by Rule 2. Hence in this case we get $|N_3(v,w) \cap V(R)| \leq 14 \cdot 7 + 1 = 99$.

Summarizing, using that $V(R) = \{v,w\} \cup (N_1(v,w) \cup N_2(v,w) \cup N_3(v,w)) \cap V(R)$, we get that $|V(R)| \leq 2 + 4 + 54 + \max(57, 3, 77, 99) = 159$. □

We finally have all the ingredients in order to prove Theorem 1.

Proof of Theorem 1. Let G be a plane graph and let G' be the reduced graph. According to Lemmas 2 and 4, G admits a TDS of size at most k if and only if G' admits one. According to Lemmas 3 and 5, G is reduced in time $O(n^6)$, so Rules 1 and 2 define a polynomial-time reduction. According to Propositions 1, 2, and 3, if G' admits a TDS of size at most k then G' has size at most $159 \cdot (3 \cdot k - 6) + 218 \cdot k \leq 695 \cdot k$. So if G' has size greater than $695 \cdot k$ we can safely reject the instance, and otherwise the reduced graph defines the desired linear kernel. □

6 Further research

As we showed in Section 2, Total Dominating Set satisfies the general conditions of the meta-theorem of [3], and therefore we know that there *exists* a linear kernel on graphs of bounded genus. Finding an *explicit* linear kernel on this class of graphs seems a feasible but involved generalization of our results. It is worth mentioning that neither (Connected) Dominating Set nor Total Dominating Set satisfy the general conditions of the meta-theorems for H-minor-free graphs [9] and H-topological-minor-free graphs [23]. But it has been recently proved by Fomin *et al.* that (Connected) Dominating Set has a linear kernel on H-minor-free graphs [11] and, more generally, in H-topological-minor-free graphs [10]. Is it also the case of Total Dominating Set?

It would be interesting to use our techniques to obtain linear kernels on planar graphs for other domination problems. For instance, Independent Dominating Set admits a kernel of size $O(jk^i)$ graphs which exclude $K_{i,j}$ as an induced subgraph [26]; this implies, in particular, a cubic kernel on planar graphs. It is proved in [3] that Independent Dominating Set has a polynomial kernel on graphs of bounded genus, even if the problem does not have FII. The existence of a linear kernel on planar graphs remains open. There are other variants of domination problems for which the existence of polynomial kernels on sparse graphs has not been studied yet, like Acyclic Dominating Set [5, 17] or α-Total Dominating Set [20].

Concerning approximation, it may be possible that Total Dominating Set admits a PTAS on planar graphs, as it is the case of many other graph optimization problems [2]. See also [28] for other results about the computational complexity of Total Dominating Set.

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